On the development of mathematical competencies of students in the construction and solution of complex inequalities

Sobre el desarrollo de competencias matemáticas de los estudiantes en la construcción y solución de desigualdades complejas

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ABSTRACT:
In the article, the problem of the development of mathematical competencies of students in constructing and solving complex inequalities is studied. Based on the analysis of psychological and pedagogical literature, the conditions for the development of mathematical competencies of students in the construction and solution of complex inequalities are revealed. The methodical instruments contributing to the development of mathematical competencies of students in the construction and solution of complex inequalities are analyzed. The functions of constructing and solving complex inequalities in the development of mathematical competencies of students are discovered. Mathematical methods have been developed, which provide a level-by-level understanding of the educational material through enrichment of the

RESUMEN:
En este artículo se estudia el problema del desarrollo de las competencias matemáticas de los estudiantes en la construcción y resolución de desigualdades complejas. Basándose en el análisis de la literatura psicológica y pedagógica, se revelan las condiciones para el desarrollo de competencias matemáticas de los estudiantes en la construcción y solución de desigualdades complejas. Se analizan los instrumentos metódicos que contribuyen al desarrollo de competencias matemáticas de los estudiantes en la construcción y solución de desigualdades complejas. Se descubren las funciones de construir y resolver las desigualdades complejas en el desarrollo de las competencias matemáticas de los estudiantes. Se han desarrollado métodos matemáticos, que proporcionan una comprensión de nivel por nivel del material
1. Introduction

In modern conditions, the problem of developing the mathematical competencies of students in the construction and solution of complex inequalities acquires particular urgency. This is due to the ever-increasing needs of society in competent individuals capable of posing new problems, and finding innovative solutions in the conditions of uncertainty and multiple choices (Chown (1994), Sakenov et al. (2012), Gifford (1994), Makhashova et al. (2016)). One can naturally pose a question of organizing cognitive activity of students that promotes the development of mathematical competencies as the basis for self-realization of the person in the subsequent stages of continuous education: the problem of finding opportunities for the development of mathematical competencies of students in the framework of educational activity is topical. It is suggested to develop the mathematical competencies of students with the help of specially designed tasks, the organization of independent research work, the creation of question-answer procedures, etc. The research of this kind (White (1959), Austin, & Howson (1979), Salekhova (2014), Alpyssov et al. (2016)) is mainly devoted to the formation of certain aspects of the mathematical competence of students, while, as a whole, the problem of developing the mathematical competencies of students still remains poorly developed.

In a number of psychological and pedagogical works (Omarov et al. (2016), Berkimbaev et al. (2012), Zhumasheva et al. (2016), Sharmin (2005), OECD (2012)), the competencies of a specialist are considered as an important factor in the development of mathematical competencies of students. The analysis and generalization of the practice of presentation of educational material in mathematics show that the problem of finding didactic possibilities for constructing and solving complex inequalities for the development of mathematical competencies of students in the framework of educational activity still remains open. The current situation can be characterized by the following contradictions:

1. contradictions between the importance of the problem of developing the mathematical competencies of students and the insufficiently developed methodological tools aimed at their development, in particular, in the construction and solution of complex inequalities;
2. contradictions between the high developmental possibilities of constructing and solving complex inequalities and the insufficiently developed scientific and methodological foundations of teaching mathematics with the use of different methods of constructing and solving complex inequalities that are aimed at the development of the mathematical competencies of students.

On the basis of the revealed contradictions, the analysis of philosophical, psychological and pedagogical literature, as well as the study of the work experience, the goal of the research was formulated: theoretical substantiation and development of a methodology for the development of mathematical competencies of students in constructing and solving complex inequalities.

2. Methods

The methodological bases of the research of development of mathematical competencies of students in constructing and solving complex inequalities are as follows: the system approach to studying the pedagogical phenomena; activity-based and personal-developmental approach to training; psychologically oriented theories of training; theoretical positions on the problems of intellectual and creative development of the person; the concepts of intellectual education on
the basis of enriching student’s mental experience in learning mathematics; theoretical developments in the field of formation of conceptual thinking; theoretical substantiation of the content of the mathematics course; theoretical regularities of the use of the activity approach in training; theoretical developments concerning the role of mathematical tasks in the development of personality; theory and practice of mathematical education; theory of competence approach; theory of vocational education; theory of professional competence; concepts of creating the content of education; theory of activity; theory of technological development of the educational process; theoretical foundations of the organization of educational activities. The following methods are applied: the theoretical ones (the study and analysis of psychological-pedagogical, methodical research of the problem of development of mathematical competencies of students in the process of education); the empirical ones (conversation, questioning, observation of educational activities, pedagogical experiment); the mathematical ones (statistical processing of the results of the experiment on the development of mathematical competencies of students in the construction and solution of complex inequalities).

3. Results
Mathematical competencies are a set of interrelated qualities of a person (mathematical knowledge, skills and methods of activity), defined with respect to a certain range of things and processes, and necessary for the quality productive activities in relation to them. One of the most important conditions for the development of mathematical competencies is the understanding of the educational material in the construction and solution of complex inequalities. Here it is necessary to identify the following levels: tracking, reproduction, creative understanding of the task in constructing and solving complex inequalities. Following Khinchin (2006); Clarkson (1992); Ellerton, & Clarkson (1996); and Filippov (2000), we believe that the increase in the levels of understanding of the task in the construction and solution of complex inequalities is achieved through the enrichment of the basic components of the individual mental experience: cognitive experience, reflexive experience and emotional-evaluative experience. On the basis of the theoretical analysis of the problem of developing the mathematical competencies of students in the construction and solution of complex inequalities, taking into account the psychological features and the character of the leading activity, namely, the orientation toward the object of activity, we have summarized and systematized the conditions for the development of mathematical competencies of students in the process of constructing and solving complex inequalities, based on a level-by-level organization of the process of understanding the educational material through enrichment of conceptual, reflexive and emotional-evaluative experience. Constructing and solving complex inequalities is a special didactic tool that creates conditions for the development of the mathematical competence of students (Henner (2004), Sharmin (2005), Lockhart (2014), Esmukhan, & Alpysov (2002)).

As a methodological substantiation of didactic possibilities of constructing and solving complex inequalities that are used in the training process with the aim of developing the mathematical competence of students, it is suggested to take into account:
- conditions for the development of mathematical competencies of students in the process of constructing and solving complex inequalities, in particular, the possibility of developing the mathematical competence of students on the basis of a level-by-level organization of understanding the educational material;
- indicators of development of mathematical competencies of students;
- didactic possibilities of constructing and solving complex inequalities that ensure the development of mathematical competencies of students by enriching various forms of mental experience;
- the methodology of constructing and solving complex inequalities based on three fundamental levels of understanding of educational material through enriching the conceptual, reflective,
emotionally-evaluative experience of students:

Problem 1. Let \( y_1 \) and \( y_2 \) be solutions of the equation \( y^2 - (y_1 + y_2)y + y_1 \cdot y_2 = 0 \), where \( y_1 < y_2 \). One has to compose an inequality of the form:
\[ y^2 - (y_1 + y_2 - k)y > ky - y_1 \cdot y_2, \]
where \( k \) is an arbitrary number, satisfying the condition \( k(k-k) = 0 \).
Let us rewrite this problem in the form of a theorem.

Theorem 1.
\[
\begin{cases}
  y^2 - (y_1 + y_2)y + y_1 \cdot y_2 = 0 \\
  y_1 < y_2 \\
  \forall k(k-k = 0)
\end{cases}
\Rightarrow
\begin{cases}
  y^2 - (y_1 + y_2 - k)y > ky - y_1 \cdot y_2 \\
  \text{or} \\
  y^2 - (y_1 + k)y < (y_2 - k) - y_1 \cdot y_2.
\end{cases}
\]

Analysis. 1) The generalized structural formulation of Problem 1 should be understood as follows: \( y_1 \) and \( y_2 \) are obtained from \( y^2 - (y_1 + y_2)y + y_1 \cdot y_2 = 0 \), whereas, the inequality \( y_1 < y_2 \) indicates which of the two solutions should be taken to be the smaller one, and which, the larger one, i.e., this condition introduces certainty. Here \( k \) is an arbitrary constant. By giving different values to it, we get different inequalities in the row form. This is the essence of the formulation of the Problem 1.

2) The appearance in the above expressions of an arbitrary number \( k \) aims to increase the number of terms in the expression in order to change the structure of the problem. Since zero is added to the condition of the problem, it leaves unaltered the solutions of the inequality. However, changing the structure of zero, one can obtain tasks with different structures.

Let us particularize Theorem 1.

Theorem 1.1 \( y^2 - 2y - 3 = 0 \Rightarrow y^2 - (y_1 + y_2)y + y_1 \cdot y_2 = 0 \).

In the structure of Theorem 1.1, an idea is clearly formulated that for the construction of inequality it is necessary to define various concrete equations. It is then that different inequalities with different solutions are constructed. We use \( k \) to change the structure of the equation. However, for the same equation we get different inequalities, having identical solutions.

\[ y^2 - 2y - 3 = 0 \Rightarrow y_{1,2} = 1 \pm \sqrt{1+3} = 1 \pm 2; \quad y_1 = -1; \quad y_2 = 3. \]

Let us give the value \( k = 5 \) to the parameter \( k \) and construct the inequality:
\[ y^2 + 3y > 5y + 3. \] If \( k = -7 \), then we arrive at the inequality: \[ y^2 - 9y > 3 - 7y. \]

Now we show that these two inequalities have the same solutions. To apply the inverse action method, we separate the complete square of a binomial, moving all the terms of the inequality to one side.

The solution of the first inequality:
\[
\begin{align*}
  y^2 - 2y - 3 > 0 & \Rightarrow (y-1)^2 - 3 - 1 > 0 \Rightarrow (y-1)^2 > 4 \\
  \Rightarrow \begin{cases}
    y-1 > 2 \\
    y-1 < -2
  \end{cases} & \Rightarrow \begin{cases}
    y > 3 \\
    y < -1
  \end{cases}
\end{align*}
\]

The answer: \((\infty, -1) \cup (3, +\infty)\).

The solution of the second inequality:
\[
\begin{align*}
  y^2 - 9y > 3 - 7y & \Rightarrow y^2 - 2y - 3 > 0 \\
  \Rightarrow \begin{cases}
    y-1 > 2 \\
    y-1 < -2
  \end{cases} & \Rightarrow \begin{cases}
    y > 3 \\
    y < -1
  \end{cases}
\end{align*}
\]

The answer: \((\infty, -1) \cup (3, +\infty)\).

Despite the structural difference, both inequalities have the same solutions. Next, this regularity, that is, the sameness of the solution of the quadratic inequality and the transformed inequality, will be established.
Theorem 1.2

\[ y^2 - (y_1 + y_2) y + y_1 \cdot y_2 < 0 \]
\[ y_1 < y_2 \]
\[ \forall k (k - k = 0) \]

\[ \Rightarrow y^2 - (y_1 + k) y < (y_2 - k) - y_1 \cdot y_2 \]

First, we solve the inequality \( y^2 - (y_1 + y_2) y + y_1 \cdot y_2 < 0 \). The process of solving the inequality is simplified if we solve it by the inverse action method. To do this, we separate the complete square of a binomial.

\[ y^2 - (y_1 + y_2) y + y_1 \cdot y_2 < 0 \Rightarrow y^2 - 2 \left( \frac{y_1 + y_2}{2} \right) y + y_1 \cdot y_2 < 0 \Rightarrow \]
\[ \Rightarrow \left( y - \frac{y_1 + y_2}{2} \right)^2 + \frac{4y_1 \cdot y_2 - (y_1 + y_2)^2}{4} < 0 \Rightarrow \]
\[ \Rightarrow \left( y - \frac{y_1 + y_2}{2} \right)^2 + \frac{4y_1 \cdot y_2 - y_1^2 - 2y_1 \cdot y_2 - y_2^2}{4} < 0 \Rightarrow \]

To solve the latter inequality by the inverse action method, the expression containing the unknown must be left in one side of the inequality, and the expression containing known numbers must be carried to the other side of the inequality.

\[ \left( y - \frac{y_1 + y_2}{2} \right)^2 < \frac{(y_2 - y_1)^2}{2} \]

Putting \( y - \frac{y_1 + y_2}{2} > 0 \), we have

\[ \left( y - \frac{y_1 + y_2}{2} \right) < \frac{y_2 - y_1}{2} \text{ or } y < \frac{y_1 + y_2}{2} + \frac{y_2 - y_1}{2} \Rightarrow y < y_2. \]

Setting \( y - \frac{y_1 + y_2}{2} < 0 \), and \( y_2 - y_1 > 0 \), \( \sqrt{y_2 - y_1} = -(y_2 - y_1) = y_1 - y_2 \), we have

\[ -\left( y - \frac{y_1 + y_2}{2} \right) < \frac{y_2 - y_1}{2} \text{ or } y > \frac{y_1 + y_2}{2} - \frac{y_2 - y_1}{2} \Rightarrow y > y_1. \]

Thus, the solution of the inequality can be written as follows: \( x \in (y_1, y_2) \).

Now we solve the inequalities \( y^2 - (y_1 + k) y < (y_2 - k) - y_1 \cdot y_2 \).

\[ y^2 - (y_1 + k) \cdot y < (y_2 - k) \cdot y - y_1 \cdot y_2 \Rightarrow y^2 - y_1 \cdot y - k \cdot y < y_2 \cdot y - ky - y_1 \cdot y_2 \Rightarrow \]
\[ \Rightarrow y^2 - y_1 \cdot y - y_2 \cdot y + y_1 \cdot y_2 < 0 \Rightarrow \]
\[ y^2 - (y_1 + y_2) y + y_1 \cdot y_2 < 0. \]

The solution of the last inequality is known and is represented by the interval \( (y_1, y_2) \).

Thus, it is proved that both inequalities have the same solutions. Moreover, the signs of the quadratic inequality and the signs of the solution of the quadratic inequality coincide:

\[ y^2 - (y_1 + y_2) y + y_1 \cdot y_2 < 0 \Rightarrow y_1 < y < y_2. \]

Arguing in a similar fashion, we see that the above-established regularities hold also for another sign of the inequality contained in the quadratic inequalities, which we can write in an abbreviated form as follows:

\textbf{Theorem 1.3}
4. Discussion

The enrollment of trainees at the research stage of the experiment was 71 people; the sample size at the final stage of the forming experiment was 71 people. At the first stage of the experiment, some primary research was conducted concerning the development of mathematical competencies of students in constructing and solving complex inequalities. The

\[
\begin{align*}
\quad & y^2 - (y_1 + y_2) y + y_1 \cdot y_2 > 0 \\
\quad & y_1 < y_2; y > y_1; y > y_2 \\
\quad & \forall k(k - k = 0)
\end{align*}
\]

\[
\Rightarrow y^2 - (y_1 + k)y \geq (y_2 - k) - y_1 \cdot y_2, y_1 < y, y > y_2.
\]

These formulas show: 1) that in the first line, there is the process of converting the standard inequality with the help of an arbitrary constant to another inequality; 2) the sameness of their solutions; 3) preservation of signs of the standard inequality in the solutions. In the generalized notation, all these patterns are perceived in a complex manner by the eyes, and the transition from one form of the inequality to its another form is not particularly difficult. If we now introduce the standard logarithmic function into the transformed inequality, then we obtain logarithmic inequalities. If we leave the values of \(k\) unchanged, but only change the standard equation or inequalities, then we obtain a transition to a standard quadratic algebraic inequality; the transition allows us immediately writing down its solutions and then proceeding to the solution of the standard logarithmic inequality. Let us show this by examples.

Example 1. One has to solve the inequalities

\[
\log_2^2(2x - 1) + \log_2(2x - 1) > 5 \log_2(2x - 1) + 12 \Rightarrow a ? \quad x ? \quad c ?
\]

Solution. According to the algorithm described above, we first define the standard logarithmic function. The standard logarithmic function is the logarithm of the sum of algebraic functions. In the inequality, this function is contained in an open form. Therefore, we denote it by

\[
y = \log_2(2x - 1).
\]

Then we obtain the algebraic inequality:

\[
y^2 + y > 5y + 12.
\]

We reduce it to the standard quadratic inequality: \(y^2 - 4y - 12 > 0\). Solving this quadratic equation, we find: \(y_1 = -2; y_2 = 6\). According to the algorithm below, the sign of the solution of the inequality must coincide with the sign of the inequality itself. Thus, we have: \(y < -2; y > 6\).

The sign of the solution of the inequality, according to the algorithm below, must coincide with the sign of the inequality itself. So, we have: \(y < -2; \text{ and } y > 6\).

Substituting the logarithmic functions into these inequalities, we obtain

\[
\log_2(2x - 1) < -2 \text{ and } \log_2(2x - 1) > 6.
\]

We solve them by the method of reverse actions. Since the base of the logarithm is greater than unity, the inequality sign is preserved. We have:

\[
\log_2(2x - 1) < -2 \Rightarrow 2^{\log_2(2x - 1)} < 2^{-2} \Rightarrow 2x - 1 < \frac{1}{4} \Rightarrow
\]

\[
\Rightarrow 2x < 1 + \frac{1}{4} = \frac{5}{4} \Rightarrow x < \frac{5}{8}.
\]

\[
\log_2(2x - 1) < -2 \Rightarrow 2^{\log_2(2x - 1)} < 2^6 \Rightarrow 2x - 1 > 64 \Rightarrow 2x > 65 \Rightarrow x > 32.5.
\]

Thus, the solution of the logarithmic inequality is the union of intervals: \((-\infty, 5/8) \cup (32.5; +\infty)\).
results of performing the selected tasks showed the inability of students to actualize their knowledge when constructing and solving complex inequalities, using common methods and ideas, demonstrating the diversity and flexibility of knowledge (Table 1).

Table 1. Indicators of the development of mathematical competencies of students, %

<table>
<thead>
<tr>
<th>Indicator type</th>
<th>The skill to see the problem</th>
<th>Flexibility</th>
<th>Originality</th>
<th>Emotionality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>41</td>
<td>11</td>
<td>16</td>
<td>7</td>
</tr>
<tr>
<td>Final</td>
<td>59</td>
<td>28</td>
<td>43</td>
<td>33</td>
</tr>
</tbody>
</table>

The experimental (E, 35 students) and the control (C, 36 students) groups were formed. In the experimental groups, the mathematical methods developed by us, providing a level-by-level understanding of the educational material through enrichment of the conceptual, reflexive and emotional-evaluative experience of students, as well as the methodology of constructing and solving complex inequalities were used. It turned out that, with approximately the same results before the beginning of the experiment, after its completion, the experimental groups had better results in terms of the development of mathematical competence: flexibility, originality and swiftness (Figure 1).

Figure 1. Results of the development of mathematical competencies of students in the construction and solution of complex inequalities: flexibility, originality and swiftness in the experimental (E) and control (C) groups

Comparing the results of the experiment on the development of mathematical competencies of students in the construction and solution of complex inequalities, we can claim that the use of constructing and solving complex inequalities based on a level-by-level understanding of the educational material and the orientation toward enriching the conceptual, reflexive and emotional-evaluative experience of students allows creating conditions for improving the quality of mathematical training of students, contributing to the development of their mathematical competence.

5. Conclusion

Thus, during the research, the role of constructing and solving complex inequalities, their influence on the level of understanding of the educational material and, as a consequence, on the development of mathematical competencies of students, are justified and established. A
sequence of constructing and solving complex inequalities is developed, and, for the first time, the foundation is the orientation toward enriching the conceptual, reflective and emotional-evaluative experience of students. The conditions for the development of mathematical competencies of students in the process of teaching mathematics are substantiated; in particular, the statement is formulated about the possibility of developing mathematical competencies of students on the basis of a level-by-level organization of the process of understanding the educational material. The didactic possibilities of constructing and solving complex inequalities are substantiated, ensuring the effectiveness of the process of developing mathematical competencies of students by enriching various forms of mental experience. We developed and implemented ways of constructing and solving complex inequalities on the basis of a level-by-level understanding of the material and enrichment of various forms of students' mental experience. The methodology of the organization of the educational process with the use of constructing and solving complex inequalities and a set of methods for organizing the activities of students in the use and construction of complex inequalities is introduced, which contributes to improving the efficiency of the process of developing the mathematical competence of students and developing the qualities of mathematical competence: swiftness, originality, flexibility, initiative and reflexivity. The methodology for developing the mathematical competencies of students in the construction and solution of complex inequalities is recommended when developing training programs at the higher education institution.

References


